An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 45 (2003) 1227–1233
www.elsevier.nl/locate/camwa

The Stability of Partial Difference Systems with Retarded Arguments

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Abstract—For certain partial difference systems with retarded arguments, sufficient conditions for the existence of decaying solutions are given. The stability of these systems is discussed also.

Keywords—Difference equations, Delay varying in time, Decaying solution, Stability.

1. INTRODUCTION

Recently, the qualitative analysis of difference equations has received much attention. In particular, the stability and oscillation of partial difference equations have been investigated intensively; see the recent monographs [1–4].

The stability and asymptotic behavior of solutions of the difference equations of the form

$$x(t) = \sum_{k=1}^{N} A_k x(t - r_k) \tag{1.1}$$

have been studied in the past years, where $x \in R^n$, A_k is an $n \times n$ matrix. $0 \le r_k \le r$, for $k = 1, 2, \dots, N$. In the scalar case, Melvin showed that a necessary and sufficient condition for the stability of (1.1) is $\sum_{k=1}^{N} |A_k| < 1$, $A_k \in R$. Avellar, Garcia and Marcanato [5] extended this result to the delays r_k and coefficients A_k in (1.1) varying in time. That is,

$$x(t) = \begin{cases} \sum\limits_{k=1}^{N} A_k(t) x(t - r_k(t)), & t \ge 0, \\ \varphi(t), & -r \le t \le 0, \end{cases}$$

where $r_k : [0, \infty)$, $A_k : [0, \infty) \to R$ are real continuous functions.

In this paper, we consider the system of partial difference equations

$$Z(x, y) = \begin{cases} \sum\limits_{k=1}^{N} A_k(x, y) Z(x - p_k(x), y - q_k(y)), & x, y \in \Omega_0, \\ \varphi(x, y), & x, y \in \Omega_2, \end{cases} \tag{1.2}$$

This work is supported by NNSF of China.

PII: S0898-1221(03)00092-0

where $p_k : [0,\infty) \to R_+$, $q_k : [0,\infty) \to R_+$, and $p_k(.), q_k(.)$ are both continuous functions. $Z, \varphi \in R^n$, $A_k : \Omega_0 \to R^{n\times n}$, $k = 1,2,\dots,N$, are real continuous functions, and

$$\Omega_0 = \{(x,y) \mid x \geq 0,\ y \geq 0\},$$
$$\Omega_1 = \{(x,y) \mid x \geq -p,\ y \geq -q\},$$
$$\Omega_2 = \Omega_1 \setminus \Omega_0,$$

where $p > 0$, $q > 0$.

Let

$$p(x) = \max_{1\leq k\leq N} p_k(x), \qquad x \geq 0,$$
$$q(y) = \max_{1\leq k\leq N} q_k(y), \qquad y \geq 0.$$

We assume that $p(x), q(y)$ satisfies: $p(x) \leq x + p$, $q(y) \leq y + q$, where $x, y \geq 0$. Given a function $\varphi(x,y) \in R^n$ on Ω_2, it is easy to see that the initial value problem (1.2) has a unique solution $Z(x,y)$ on Ω_0. Our purpose is to obtain some sufficient conditions for the stability of (1.2). For any $H > 0$, let

$$S_H = \{\|\varphi\|_{\Omega_2} < H\}.$$

Similar to the scalar equation [3], we give the following definitions.

DEFINITION 1.1. *Equation (1.2) is said to be stable if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $\varphi \in S_\delta$, the corresponding solution $Z(x,y)$ of (1.2) satisfies*

$$\|Z(x,y)\| < \epsilon, \qquad (x,y) \in \Omega_0.$$

DEFINITION 1.2. *Equation (1.2) is said to be asymptotically stable in the large if it is stable, and at the same time for every solution $Z(x,y)$ with the initial function $\varphi(x,y)$, which satisfies $\sup_{(x,y)\in\Omega_2} \|\varphi(x,y)\| = c$, c is a positive constant, satisfies that: $\|Z(x,y)\| \to 0$, as $\min(x,y) \to +\infty$.*

DEFINITION 1.3. *Equation (1.2) is said to be exponentially asymptotically stable if for any $\delta > 0$, there exists a real number $r \in (0,1)$ such that $\varphi \in S_\delta$ implies that*

$$\|Z(x,y)\| \leq \delta r^{c\min(x,y)}, \qquad c > 0, \quad (x,y) \in \Omega_0.$$

2. SOME FUNDAMENTAL RESULTS

To prove our results, we need a modified version of the Darbo fixed-point theorem [3].

LEMMA 2.1. *Let c be a nonempty, bounded, convex, and closed subset of a Banach space E. If $F : c \to c$ is a μ-contraction, then F has at least one fixed point in c and the set $\operatorname{Fix} F = \{x \in c : Fx = x\}$ belongs to the $\ker \mu$.*

REMARK 2.1. Noting set $\operatorname{Fix} F$ with K, it is easy to see that $\mu(K) = \mu(FK) = 0$. Denote by $C_0 = C(\Omega_1, R^n)$ the space of bounded continuous functions on Ω_1 with the norm $\|z\|_{\Omega_1} = \{\sup \|z(x,y)\| : (x,y) \in \Omega_1\} < \infty$. So C_0 is a Banach space.

For C_H, an arbitrary nonempty and bounded subset of C_0, $\|z\|_{\Omega_1} \leq H$, if for any $T > 0$, $\epsilon > 0$, $P = (x_1, y_1)$, $Q = (x_2, y_2) \in [-p,\infty) \times [-q,\infty)$, let us denote

$$w_\epsilon^T(z) = \{\sup \|z(x_1,y_1) - z(x_2,y_2)\| : P,\ Q \in [-p,T] \times [-q,T],\ \|P - Q\| \leq \epsilon\},$$
$$w_\epsilon^T(C_H) = \{\sup w_\epsilon^T(z) : z \in C_H\},$$
$$w_o^T(C_H) = \lim_{\epsilon\to 0} w_\epsilon^T(C_H),$$
$$w_o(C_H) = \lim_{T\to\infty} w_o^T(C_H),$$
$$a_o(C_H) = \lim_{T\to\infty} \sup_{z\in C_H} \{\sup \|z(x_1,y_1)\|,\ P \in [T,\infty) \times [T,\infty)\},$$
$$\mu(C_H) = w_o(C_H) + a_o(C_H).$$

Similar to the related result in [6], it is not difficult to prove the following conclusion.

LEMMA 2.2. *The function $\mu(C_H)$ is the sublinear measure of noncompactness in the space C_0.*

3. MAIN RESULTS

THEOREM 3.1. *Suppose the following conditions hold:*

(i) $r = \sup_{x\ge 0, y\ge 0} \sum_{k=1}^{N} \|A_k(x,y)\| < 1$; $A_K(x,y)$ *is continuous*, $(x,y) \in \Omega_0$;
(ii) $\lim_{x\to\infty} x - p(x) = \infty$;
(iii) $\lim_{y\to\infty} y - q(y) = \infty$.

Then for every given $\varphi(x,y)$, *such that* $\sup \|\varphi(x,y)\| = c < +\infty$, $(x,y) \in \Omega_2$, *the corresponding solution* $Z(x,y)$ *of (1.2) satisfies* $\|Z(x,y)\| \to 0$, *as* $\min(x,y) \to +\infty$.

PROOF. For any $M > 0$, let $h_M = \{z \in C_0 : z(x,y) = \varphi(x,y),\ (x,y) \in \Omega_2 \text{ and } \|z\|_{\Omega_1} \le M\}$, and $F : h_M \to C_0$ is the map given by

$$(Fz)(x,y) = \begin{cases} \sum\limits_{k=1}^{N} A_k(x,y) z(x - p_k(x), y - q_k(y)), & x,y \in \Omega_0, \\ \varphi(x,y), & (x,y) \in \Omega_2. \end{cases}$$

First, we should verify $F(h_M) \subseteq h_M$. For all $z \in h_M$, we have

$$\begin{aligned} \|(Fz)(x,y)\| &= \left\| \sum_{k=1}^{N} A_k(x,y) z(x - p_k(x), y - q_k(y)) \right\| \\ &\le \sum_{k=1}^{N} \|A_k(x,y)\| \|z(x - p_k(x), y - q_k(y))\| \\ &\le r\|z\|_{\Omega_1}. \end{aligned}$$

Therefore, $\|Fz\|_{\Omega_1} \le \|z\|_{\Omega_1} \le M$. That is, $F(h_M) \subseteq h_M$.

In a similar way, we obtain

$$\|Fz_1 - Fz_2\| \le \|z_1 - z_2\|.$$

Hence, F is continuous.

Next, we should verify $\mu(Fh_M) < \mu(h_M)$. For any $z \in h_M$, $T \ge 0$, $x,y \in [T,\infty) \times [T,\infty)$, we can get that: $a_0(Fh_M) \le r a_0(h_M)$. Now, let us take $P = (x_1,y_1)$, $Q = (x_2,y_2)$, $T > 0$, $P,Q \in (0,T) \times (0,T)$. Since $A_k(x,y)$ is a continuous function, so for any $\varepsilon > 0$, there is $\delta > 0$, such that if $\|P - Q\| \le \delta$, we have $\|A_k(x_1,y_1) - A_k(x_2,y_2)\| \le r\varepsilon/M$. Hence,

$$\begin{aligned} &\|(Fz)(x_1,y_1) - (Fz)(x_2,y_2)\| \\ &\quad = \left\| \sum_{k=1}^{N} A_k(x_1,y_1) z(x_1 - p_k(x_1), y_1 - q_k(y_1)) - \sum_{k=1}^{N} A_k(x_2,y_2) z(x_2 - p_k(x_2), y_2 - q_k(y_2)) \right\| \\ &\quad \le \sum_{k=1}^{N} \|A_k(x_1,y_1)\| \|z(x_1 - p_k(x_1), y_1 - q_k(y_1)) - z(x_2 - p_k(x_2), y_2 - q_k(y_2))\| \\ &\qquad + \sum_{k=1}^{N} \|A_k(x_2,y_2) - A_k(x_1,y_1)\| \|z(x_2 - p_k(x_2), y_2 - q_k(y_2))\| \\ &\quad \le \sup \sum_{k=1}^{N} \|A_k(x_1,y_1)\| \sup \|z(x_1 - p_k(x_1), y_1 - q_k(y_1)) - z(x_2 - p_k(x_2), y_2 - q_k(y_2))\| + r\varepsilon \\ &\quad \le r(\varepsilon + \sup \|z(x_1 - p_k(x_1), y_1 - q_k(y_1)) - z(x_2 - p_k(x_2), y_2 - q_k(y_2))\|). \end{aligned}$$

So, we have

$$\begin{aligned} w_\epsilon^\top(Fh_M) &\le r\left(\epsilon + w_\epsilon^\top(h_M)\right), \\ w_o^\top(Fh_M) &= \lim_{\epsilon\to 0} r w_\epsilon^\top(Fh_M) \\ &\le r w_o^\top(h_M). \end{aligned}$$

Since $\mu(h_M) = w_o^\top(h_M) + a_0(h_M)$, $\mu(Fh_M) = w_o^\top(Fh_M) + a_0(Fh_M)$, we can obtain

$$\mu(Fh_M) \le r\mu(h_M),$$

which means that F is μ-contraction, and by Lemma 2.1, F has a fixed point $z \in h_M$. It is easy to see that $z(x,y)$ is a solution of (1.2).

Since $z \in K$, $\mu(K) = 0$, we get that $a_0(K) = 0$. That is, $\|z(x,y)\| \to 0$, as $\min(x,y) \to +\infty$. The proof is complete.

In Theorem 3.1, we obtain sufficient conditions of the attractivity of the solution of (1.2). In order to reach the conclusion that (1.2) is asymptotically stable in the large, we need to prove that (1.2) is stable.

THEOREM 3.2. *Assume the conditions of Theorem 3.1 hold. Then for every given $\varphi(x,y)$, such that* $\sup\|\varphi(x,y)\| = c < +\infty$, $(x,y) \in \Omega_2$, *the solution $Z(x,y)$ of (1.2) satisfies: $\|Z(x,y)\| \le c$, $(x,y) \in \Omega_0$. Therefore, (1.2) is stable.*

PROOF. First we define a sequence of sets S_i in Ω_0 as follows.

For a point $(x,y) \in \Omega_0$, if $(x - p_k(x), y - q_k(y)) \in \Omega_2$, $k = 1,\dots,n$, then $(x,y) \in S_1$.

And for another point $(x,y) \in \Omega_0 \setminus S_1$, if $(x - p_k(x), y - q_k(y)) \in \Omega_2 \cup S_1$, $k = 1,\dots,n$, then $(x,y) \in S_2$.

Step by step, we get a series of sets $S_1, S_2, S_3, \dots$. We shall show that $\Omega_0 = \bigcup_{i=1}^\infty S_i$.

In fact, because $p_k(x), q_k(y)$ are both continuous and for any arbitrary point $(x_1,y_1) \in \Omega_0$, there exist two constants $a > 0$, $b > 0$ such that: $p_k(x) \ge a$, $q_k(y) \ge b$, $0 \le x \le x_1$, and $0 \le y \le y_1$, it is sure that

$$(x_1,y_1) \in \bigcup_{i=1}^{\max([x_1/a],[y_1/b])+1} S_i.$$

It is easy to see that

$$\|Z(x,y)\| \le \sum_{k=1}^n \|A_k(z,y)\| \|z(x - p_k(x), y - q_k(y))\|.$$

Therefore, $\sup_{(x,y)\in S_1} \|Z(x,y)\| \le c$.

In a similar way, we have

$$\sup_{(x,y)\in S_2} \|Z(x,y)\| \le \max\left(\sup_{(x,y)\in S_1} \|Z(x,y)\|, c\right) \le c.$$

By the induction, we have

$$\sup_{(x,y)\in S_i} \|Z(x,y)\| \le c, \qquad i = 1,2,3,\dots.$$

The proof is complete.

Combining Theorems 3.1 and 3.2, we have the following corollary.

COROLLARY 3.1. *Assume that the assumptions of Theorem 3.1 hold. Then (1.2) is asymptotically stable in the large.*

About the exponential asymptotic stability of (1.2), we have the following result.

THEOREM 3.3. *Suppose* $r = \sup_{x\geq 0,\, y\geq 0} \sum_{k=1}^{N} \|A_k(x,y)\| < 1$. *If there exist positive numbers* a *and* A *such that* $0 < a \leq p_k(x)$, $0 < a \leq q_k(y)$, $1 \leq k \leq N$ *and* $p(x) \leq A$, $q(y) \leq A$, $(x,y) \in \Omega_0$, *then for every given* $\varphi(x,y) \in R^n$, *with* $\|\varphi\|_{\Omega_2} = \sup_{(x,y)\in\Omega_2} \|\varphi(x,y)\| = c < +\infty$, *(1.2) has a unique solution* $Z(x,y)$ *such that*

$$\|Z(x,y)\| \leq cr^{[\min(x,y)/A]}, \qquad (x,y) \in \Omega_0, \tag{3.1}$$

where $[\cdot]$ *denoted the greatest integer less than or equal to* $\min(x,y)/A$.

PROOF. For any given $\varphi(x,y)$, it is easy to see that (1.2) has a unique solution $Z(x,y)$.

First, we assume that $x \leq a$, or $y \leq a$: Because $p_k(x) \geq a$, $q_k(y) \geq a$, $1 \leq k \leq N$, we have $(x - p_k(x), y - q_k(y)) \in \Omega_2$, and therefore,

$$\begin{aligned} \|Z(x,y)\| &= \left\| \sum_{k=1}^{N} A_k(x,y) Z(x - p_k(x), y - q_k(y)) \right\| \\ &\leq \sum_{k=1}^{N} \|A_k(x,y)\| \max_{1\leq k\leq N} \|Z(x - p_k(x), y - q_k(y))\| \\ &\leq r \max_{1\leq k\leq N} \|Z(x - p_k(x), y - q_k(y))\| \\ &\leq r\|\varphi\|_{\Omega_2} \\ &\leq cr. \end{aligned} \tag{3.2}$$

Because $\min(x - p_k(x), y - q_k(y)) \leq 0$, inequality (3.1) holds.

Next, we assume for a positive integer m, (3.1) holds for $x \leq (m-1)a$, or $y \leq (m-1)a$; i.e.,

$$\|Z(x,y)\| \leq cr^{[\min(x,y)/A]}, \qquad x \leq (m-1)a, \quad \text{or} \quad y \leq (m-1)a.$$

For $(x,y) \in \{(x,y) \mid x > (m-1)a,\ y > (m-1)a\} \setminus \{(x,y) \mid x > ma,\ y > ma\}$, it is easy to see that $(x - p_k(x)) \leq (m-1)a$, or $(y - q_k(y)) \leq (m-1)a$. From (3.2), we have

$$\begin{aligned} \|Z(x,y)\| &\leq r \max_{1\leq k\leq N} \|Z(x - p_k(x), y - q_k(y))\| \\ &\leq r \max_{1\leq k\leq N} cr^{[\min(x-p_k(x),y-q_k(y))/A]} \\ &\leq c \max_{1\leq k\leq N} r^{[\min(x-p_k(x),y-q_k(y))/A+1]}. \end{aligned} \tag{3.3}$$

Since $p_k(x) \leq p(x) \leq A$, $q_k(x) \leq q(x) \leq A$, and $\min(x - p_k(x), y - q_k(x)) + A \geq \min(x,y)$, from (3.3), we get

$$\|Z(x,y)\| \leq cr^{[\min(x,y)/A]}.$$

By the induction, we obtain that (3.1) holds on Ω_0. The proof is complete.

By Definition 1.3, we obtain the following corollary.

COROLLARY 3.2. *Under conditions of Theorem 3.3, (1.2) is exponentially asymptotically stable.*

Consider the scalar partial difference equation

$$\begin{gathered} z(x+1, y+1) = a(x,y)z(x+1,y) + b(x,y)z(x,y+1) + p(x,y)z(x-s, y-t), \\ x \geq 0, \quad y \geq 0, \end{gathered} \tag{3.4}$$

where $s, t > 0$ *are constants. The stability of (3.4) for the discrete arguments has been investigated in [3,4].*

Similar to the proof of Theorem 3.3, we can obtain the following result about the attractivity of solutions of (3.4).

COROLLARY 3.3. *Assume that* $|a(x,y)| + |b(x,y)| + |p(x,y)| \leq r < 1$. *Then for any* $\varphi(x,y)$, $(x,y) \in \Omega$, $\Omega = \{(x,y) \mid x \geq -s,\ y \geq -t\} \setminus \{(x,y) \mid x \geq 0,\ y \geq 0\}$ *satisfies* $\sup_{(x,y)\in\Omega} |\varphi(x,y)| = c < +\infty$, *equation (3.4) has a unique solution* $z(x,y)$ *with* $|z(x,y)| \to 0$, *as* $\min(x,y) \to +\infty$. *I.e., under the assumption of Corollary 3.3, equation (3.4) is attractive.*

If we put more conditions on the initial function, then we can obtain stronger results.

THEOREM 3.4. *Suppose* $r = \sup_{x\geq 0, y\geq 0} \sum_{k=1}^{N} \|A_k(x,y)\| < 1$. *If there exist positive numbers* a *and* A *such that* $0 < a \leq p_k(x) + q_k(y)$, $1 \leq k \leq N$, *and* $p(x) + q(y) \leq A$, $(x,y) \in \Omega_0$, *then for every given* $\varphi(x,y) \in R^n$, *with* $\|\varphi(x,y)\| \leq cr^{(x+y)/A}$ *on* Ω_2, *(1.2) has a unique solution* $Z(x,y)$ *such that*

$$\|Z(x,y)\| \leq cr^{[(x+y)/A]}, \qquad (x,y) \in \Omega_0, \tag{3.5}$$

where $[\cdot]$ *denoted the greatest integer less than or equal to* $(x+y)/A$.

PROOF. For any given $\varphi(x,y)$, it is easy to see that (1.2) has a unique solution $Z(x,y)$.

First, we assume that $x+y \leq a$. Because $p_k(x)+q_k(y) > a$, $1 \leq k \leq N$, we have $(x-p_k(x), y-q_k(y)) \in \Omega_2$, and therefore,

$$\begin{aligned}
\|Z(x,y)\| &= \left\| \sum_{k=1}^{N} A_k(x,y) Z(x-p_k(x), y-q_k(y)) \right\| \\
&\leq \sum_{k=1}^{N} \|A_k(x,y)\| \max_{1\leq k\leq N} \|Z(x-p_k(x), y-q_k(y))\| \\
&\leq r \max_{1\leq k\leq N} \|Z(x-p_k(x), y-q_k(y))\| \\
&\leq r\|\varphi\|_{\Omega_2} \\
&\leq cr.
\end{aligned} \tag{3.6}$$

Because $[(x+y)/A] = 0$, inequality (3.5) holds.

Next, we assume for a positive integer m, (3.5) holds for $x+y \leq (m-1)a$; i.e.,

$$\|Z(x,y)\| \leq cr^{[(x+y)/A]}, \qquad x+y \leq (m-1)a.$$

For $(m-1)a < x+y \leq ma$, $(x-p_k(x)) + (y-q_k(y)) \leq (m-1)a$, from (3.6), we have

$$\begin{aligned}
\|Z(x,y)\| &\leq r \max_{1\leq k\leq N} \|Z(x-p_k(x), y-q_k(y))\| \\
&\leq r \max_{1\leq k\leq N} cr^{[(x+y-p_k(x)-q_k(y))/A]} \\
&\leq c \max_{1\leq k\leq N} r^{[(x+y-p_k(x)-q_k(y))/A+1]}.
\end{aligned} \tag{3.7}$$

Since $p_k(x) + q_k(y) \leq p(x) + q(y) \leq A$, from (3.7), we get

$$\|Z(x,y)\| \leq cr^{[(x+y)/A]}.$$

By the induction, we obtain that (3.5) holds on Ω_0. The proof is complete.

4. EXAMPLE

Consider the system

$$Z(x,y) = A_1(x,y)Z(x-p_1(x), y-q_1(y)) + A_2(x,y)Z(x-p_2(x), y-q_2(y)), \tag{4.1}$$

where

$$Z \in R^2, \quad A_1(x,y) = \begin{pmatrix} \dfrac{\sin(xy)}{3} & 0 \\ 0 & \dfrac{-\sin(xy)}{3} \end{pmatrix}, \quad A_2(x,y) = \begin{pmatrix} 0 & \dfrac{\sin(x+y)}{3} \\ \dfrac{-\sin(x+y)}{3} & 0 \end{pmatrix}.$$

It is easy to check that

$$\|A_1\| = \|A_2\| = \frac{1}{3},$$
$$r = \|A_1\| + \|A_2\| = \frac{2}{3} < 1.$$

First, we suppose $p_1(x) = 2$, $q_1(y) = 3$, $p_2(x) = 1$, $q_2(y) = 4$. Let $a = 0.5$, $A = 4$, and

$$\Omega_0 = \{(x,y) \mid x > 0,\ y > 0\},$$
$$\Omega_1 = \{(x,y) \mid x \geq -2,\ y \geq -4\},$$
$$\Omega_2 = \Omega_1 \setminus \Omega_0.$$

From Theorem 3.3, we obtain the following conclusion.

Given any initial function $\varphi(x,y) \in R^2$, and $\|\varphi\|_{\Omega_2} = c < +\infty$, there exists a solution $Z(x,y)$ of (4.1) with

$$\|Z(x,y)\| \leq c r^{[\min(x,y)/4]}, \qquad (x,y) \in \Omega_0.$$

Next, we suppose $p_1(x) = 0.5x + 2$, $q_1(y) = \ln y - 3$, $p_2(x) = (1/3)\,x + 1$, $q_2(y) = 4$, where $p_1(x)$, $p_2(x)$, $q_1(y)$ are unbounded on Ω_0.

From Theorem 3.1, for any given function $\varphi(x,y) \in R^2$, and $\|\varphi\|_{\Omega_2} = c < +\infty$, there exists a solution $Z(x,y)$ of (4.1) with

$$\|Z(x,y)\| \longrightarrow 0, \qquad \text{when } \min(x,y) \longrightarrow +\infty.$$

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